XIV. *On finding the Values of Algebraical Quantities by converging Serieses, and demonstrating and extending Propositions given by* Pappus *and others. By* Edward Waring, *F.R.S. Professor of Mathematics at* Cambridge.

Read February 8, 1787.

SUPPOSE the roots of the equation $x^h \pm 1 = 0$ to be given, where h denotes any whole number or fraction; to find the roots or values of any given algebraical quantity, by converging infinite serieses.

1. Let the algebraical quantity be $\sqrt[n]{(\pm A)}$, then the roots of the algebraical quantity will be $A^{\frac{1}{n}} \times (\alpha + \lambda\sqrt{-1})$, $A^{\frac{1}{n}} \times (\beta + \mu\sqrt{-1})$, $A^{\frac{1}{n}} \times (\gamma + \nu\sqrt{-1})$, &c. where $\alpha + \lambda\sqrt{-1}$, $\beta + \mu\sqrt{-1}$, $\gamma + \nu\sqrt{-1}$, &c. are the roots of the equation $x^n \pm 1 = 0$; it will be $+1$ if it was $-A$, and -1 if $+A$.

2. Let the given algebraical quantity be $\sqrt[r]{(\pm\sqrt[n]{(\pm A)} \pm \sqrt[m]{(\pm B)} \pm \sqrt[p]{\pm C} \pm \text{\&c.})}$, and $\alpha + \lambda\sqrt{-1}$, $\alpha' + \lambda'\sqrt{-1}$, $\alpha'' + \lambda''\sqrt{-1}$, &c. and $\Gamma + \Delta\sqrt{-1}$ be respectively one of the roots of the equations $x^n \mp 1 = 0$, $x^m \mp 1 = 0$, $x^p \mp 1 = 0$, &c. and $x^r \mp 1 = 0$; substitute $\pm P = \pm A^{\frac{1}{n}}\alpha \pm B^{\frac{1}{n}}\alpha' \pm C^{\frac{1}{n}}\alpha'' \pm$ &c. and $\pm Q = \pm A^{\frac{1}{n}}\lambda \pm B^{\frac{1}{m}}\lambda' \pm C^{\frac{1}{m}}\lambda'' \pm$ &c. In the first place let P be greater Q, and $\pm P$ be $+P$, then will $(P \pm Q\sqrt{(-1)})^{\frac{1}{r}} =$ (P

$$\left(P^{\frac{1}{r}} - \frac{1}{r} \cdot \frac{1-r}{2r} \times \frac{Q^2}{P^{\frac{2r-1}{r}}} + \frac{1}{r} \cdot \frac{1-r}{2r} \cdot \frac{1-2r}{3r} \cdot \frac{1-3r}{4r} \times \frac{Q^4}{P^{\frac{4r-1}{r}}} - \&c. = \pm L\right) \pm \left(\frac{1}{r} \cdot \frac{Q}{P^{\frac{r-1}{r}}} - \frac{1}{r} \cdot \frac{1-r}{2r} \cdot \frac{1-2r}{3r} \cdot \frac{Q^3}{P^{\frac{3r-1}{r}}} + \frac{1}{r} \cdot \frac{1-r}{2r} \cdot \frac{1-2r}{3r} \cdot \frac{1-3r}{4r} \cdot \frac{1-4r}{5r} \times \frac{Q^5}{P^{\frac{5r-1}{r}}} - \&c. = \pm M\right) \times \sqrt{-1} = \pm L \pm M$$

$\sqrt{-1}$, in which case the two serieses $\pm$ L and $\pm$M converge, and $(\Gamma + \Delta\sqrt{-1}) \times (\pm L \pm M\sqrt{-1})$ will be a value or root of the given quantity.

In the same manner the remaining roots may be deduced.

2. Let $\pm P$ be $-P$, multiply $-P \pm Q\sqrt{(-1)}$ into -1, and it becomes $P \mp Q\sqrt{(-1)}$ a quantity of the same formula as the preceding; let $\Gamma' + \Delta'\sqrt{(-1)}$ be a root of the equation $x^r + 1 = 0$, then will $(\Gamma' + \Delta'\sqrt{-1})\,(\pm L \pm M\sqrt{-1}) = \pm H' \pm K'\sqrt{(-1)}$ be a root of the given quantity: otherwise; the root may be deduced from the above-mentioned series by substituting in it for $-(P)^{\frac{1}{r}}$ its value $P^{\frac{1}{r}} \times (-1)^{\frac{1}{r}}$, and it will become the same as the preceding.

3. Let P be less than Q, and the value of $(\pm P \pm Q\sqrt{-1})^{\frac{1}{r}}$ may be deduced from the preceding series by substituting in it $\pm Q\sqrt{-1}$ for P, and $\pm P$ for Q: otherwise, since $(\pm P \pm Q\sqrt{(-1)})^{\frac{1}{r}} = \pm\sqrt[2r]{(-1)} \times (Q \mp P\sqrt{(-1)})^{\frac{1}{r}}$, and the root of $(Q \mp P\sqrt{(-1)})^{\frac{1}{r}}$ can be deduced by the preceding method, which suppose $L' + M'\sqrt{(-1)}$; multiply this root into $H \pm \Theta\sqrt{(-1)}$, where $H + \Theta\sqrt{(-1)}$ denotes a value of the root $\pm\sqrt[2r]{(-1)}$, and the quantity resulting will be

be one value of the given quantity; the remaining values can be deduced by the fame method.

In this cafe the given quantity is refolved into a feries afcending according to the dimenfions of P, and defcending according to the dimenfions of Q; in the former cafe it was refolved into a feries afcending according to the dimenfions of Q, and defcending according to the dimenfions of P; both the feriefes affording the poffible or impoffible parts will always converge.

4. If $P = \pm Q$, then will $(\pm P \pm P\sqrt{(-1)})^{\frac{1}{r}} = P^{\frac{1}{r}} \times (\pm 1 \pm \sqrt{(-1)})^{\frac{1}{r}} = P^{\frac{1}{r}} \times 2^{\frac{1}{2r}}(\pm\sqrt{\tfrac{1}{2}} \pm \sqrt{-\tfrac{1}{2}})^{\frac{1}{r}} = P^{\frac{1}{r}} \times 2^{\frac{1}{2r}} \times \sqrt[4r]{(-1)}$; for $\sqrt[4]{(-1)} = \pm\sqrt{\tfrac{1}{2}} \pm \sqrt{-\tfrac{1}{2}}$.

4. 2. When $P = 0$, or $Q = 0$, then it becomes the firft cafe $\sqrt[n]{(\pm A)}$.

5. Let $P = Q \mp \alpha$, where α has a very fmall ratio to Q; then will $(P \pm Q\sqrt{(-1)})^{\frac{1}{r}} = (P \pm \overline{P \pm \alpha}\sqrt{(-1)})^{\frac{1}{r}} = (P \times 2^{\frac{1}{2}} \times \overline{-1}^{\frac{1}{4}} \pm \alpha\sqrt{(-1)})^{\frac{1}{r}} = P^{\frac{1}{r}} \times 2^{\frac{1}{2r}} \times \sqrt[4r]{(-1)} \pm \frac{1}{r} \times P^{\frac{1-r}{r}} \times 2^{\frac{1-r}{2r}} \times \sqrt[\frac{4r}{1-r}]{(-1)} \times \sqrt{(-1)}\,\alpha - \frac{1}{r} \times \frac{1-r}{2r} \times P^{\frac{1-2r}{r}} \times 2^{\frac{1-2r}{2r}} \times \sqrt[\frac{4r}{1-2r}]{(-1)} \times \alpha^2 \pm \frac{1}{r} \cdot \frac{1-r}{2r} \cdot \frac{1-2r}{3r} \times P^{\frac{1-3r}{r}} \times 2^{\frac{1-3r}{2r}} \times \sqrt[\frac{4r}{1-3r}]{(-1)} \times \sqrt{(-1)}\,\alpha^3 + \&c.$ In this feries the fame root of the quantity $\sqrt[4r]{(-1)}$ is always to be ufed.

6. If in the given quantity are contained more quantities of the above-mentioned kind or their roots; then, by repeating the fame operation, can be deduced the roots or values of the given quantity.

In fome cafes the impoffible part may vanifh, which may be the cafe in a quantity of the following formula, *viz.* $\sqrt[n]{a+\alpha\sqrt[2m]{-b}}+\sqrt[n]{a+\beta\sqrt[2m]{-b}}+\sqrt[n]{a+\gamma\sqrt[2m]{-b}}+$ &c. where α, β, γ, &c. denote the $2m$ roots of $\sqrt[2m]{-1}$. The general principles of difcovering the cafes in which this happens have been given in the Meditationes Algebraicæ.

The roots of the equation $x^h \pm 1 = 0$ will be found from common algebra and thefe principles, if h is not greater than 10; or, more generally, if $h = 2^l \times 3^{l'} \times 4^{l''} \ldots 10^{l^8}$, where l, l', l'' .. l^8 denote any whole numbers: or, in general, the roots of the above-mentioned equation, or even of the equation $x = \sqrt[r]{\pm L \pm M\sqrt{-1}}$, can be found from tables of fines.

The fame principles may be applied to the difcovery of the values of exponential irrational quantities.

In the Mifcel. Analy. was given, from a fubftitution invented by me and not fimilar to any before given, a refolution of equations, which contains the refolutions of all equations before given, and from which the refolutions of fome equations, not before delivered, have been added.

Part II. 1. Let an equation $A=0$ involving (r) unknown independent quantities be predicated of another equation containing the fame quantities, and the demonftration of it be required.

1ft. Reduce both the equations to equations involving independent quantities only; then reduce the two equations to one, fo that one of the above-mentioned quantities may be exterminated, and if there refults a felf-evident equation, *viz.* $A=A$, or $A-A=0$, in which the correfpondent terms deftroy each other refpectively; then the firft equation is juftly predicated of the fecond; that is, if the above-mentioned

equations

equations afford the fame value of the quantity exterminated, the propofition is true, otherwife not.

Cor. From thefe principles can be demonftrated many propofitions given by PAPPUS and others.

Ex. Let $AD = 2AC = 2x$, $DE = a$, and $EB = b$, where AD, DE, and EB, are independent quantities; if $AB \times BE = (2x + a + b) \times b = CB \times BD = (x + a + b)(a + b)$, then will $CB = x + a + b : BD = a + b :: AC \times CE = x \times (x + a) : AD \times DE = 2x \times a$.

From hence can be deduced the two equations $(b - a)\, x = a^2 + ab$ and $2a \times (x + a + b) = (a + b) \times (x + a)$; reduce thefe two equations to one, fo as to exterminate x, and there refults the felf-evident equation $(a - b) \times \frac{a^2 + ab}{b - a}(-a^2 - ab) + a^2 + ab = 0$, and confequently the propofition is true.

2. If (s) equations involving $(t + r)$ unknown and independent quantities be predicated of (t) equations involving the above-mentioned quantities: reduce the (t) equations and one of the above-mentioned (s) equations to one, fo that (t) unknown quantities may be exterminated, and if there refults a felf-evident equation, then the above-mentioned equation is juftly predicated of the (t) equations; and in the fame manner we may reafon concerning the remaining $(s - 1)$ equations.

3. 1. If one equation is juftly predicated of another, and in both the unknown quantity exterminated has only one dimenfion; then the latter equation can be predicated of the former; for in this cafe both equations have only one and the fame value of the unknown quantity exterminated.

3. 2. If the quantity exterminated has more dimenfions than one in the equations, then the propofition may not gene-

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rally be true; for the equations may have ſome roots the ſame, but not all.

Theſe obſervations may be applied to more equations.

4. From (n) given equations $a=0$, $b=0$, $c=0$, &c. can eaſily be deduced others dependent on them, by finding any direct algebraical functions of the above-mentioned equations, that is, $\varphi(a, b, c, \&c.)$, which will always $=0$; and in like manner, from the relation between any lines being given, can be deduced innumerable relations between the above-mentioned lines, and other lines dependent on them.

Part III. 1. Ratios, which are ſuppoſed greater or leſs than others, can eaſily be transformed into equations, which contain affirmative and negative quantities: for example, let the ratio $a:b$ be greater than the ratio $c:d$, then will $\frac{b}{a}=\frac{c}{d}-k$; if it be leſs, then will $\frac{b}{a}=\frac{c}{d}+k$, where k denotes an affirmative quantity; and, *vice verſâ*, if $\frac{b}{a}=\frac{c}{d}-k$, then will the ratio of $a:b$ be greater than the ratio of $c:d$, &c.

2. If one quantity (a) is affirmed to be greater than another b, for a in the given equations ſubſtitute its value $b+k$; if leſs, for a write $b-k$, where k denotes an affirmative quantity.

3. Reduce the equations, ſo as to take away their denominators, and the demonſtration of the propoſition will often very eaſily follow.

4. Let $k=\frac{P}{Q}$ and $k'=\frac{P'}{Q'}$; and if P and Q be affirmative, let P′ and Q′ be affirmative; and, *vice verſâ*, if negative, negative; then, if k be affirmative, will k' alſo be affirmative; the ſame alſo may be affirmed, if P and Q have both con-

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trary figns to P′ and Q′; but if one has the fame, and the other contrary, then will k and k' have contrary figns.

5. Let fome affirmative quantities be lefs than others, then any direct affirmative function of the former, *viz.* function, in which no negative or impoffible quantities or indexes are contained, will be lefs than the fame function of the latter. The contrary happens when the indexes are all negative, and the quantities affirmative as before: for example, let two quantities be lefs than two others, then the product of the two former will be lefs than the product of the two latter.

Cor. Hence fome quantities may often be known to be greater or lefs than others from their direct functions being greater or lefs than the fame functions of the others: for example, let $a^2 - b^2$ be an affirmative quantity, then will a be greater than b.

6. If one equation or ratio is affirmed on the fuppofition that another given one is true, reduce both the equations by the methods given above, and from the principles before delivered, the propofition will often be evident.

Hence may be deduced demonftrations to propofitions of this fort given by Pappus and others.

Ex. Let the ratio $a+b : b$ be greater than $c+d : d$, then the ratio $b : a-b$ will be lefs than $d : c-d$.

For, fince the ratio $a+b : b$ is greater than $c+d : d$, the ratio $b : a+b$ will be lefs than $d : c+d$, and confequently $\frac{a+b}{b}\left(\frac{a}{b}+1\right)=\frac{c+d}{d}\left(\frac{c}{d}+1\right)+k$, whence $\frac{a}{b}-1=\frac{c}{d}-1+k$, and $\frac{a-b}{b}=\frac{c-d}{d}+k$, and the ratio $b : a-b$ lefs than $d : c-d$.

Ex. 2. Let the ratio of $a+b : c+d$ be greater than the ratio of $a : c$, then will the ratio of $b : d$ be greater than the ratio of $a+b : c+d$. By the preceding method convert thefe ratios

tios into equations, and there reſult $\frac{c+d}{a+b}+k=\frac{c}{a}$ and $\frac{d}{b}+k'=\frac{c+d}{a+b}$; and the propoſition aſſerts, that if k be an affirmative quantity, k' will alſo be an affirmative quantity. Reduce theſe two equations, ſo as to take away their denominators, and the reſulting equations will be $ac+ad+a\times\overline{a+b}\times k=ac+bc$ and $ad+bd+\overline{a+b}\,.\,bk'=bc+bd$, whence $k=\frac{bc-ad}{a(a+b)}$ and $k'=\frac{bc-ad}{b(a+b)}$, and the propoſition is evident.

Ex. 3. Let a be greater than c, and b, and $(a+b)\times(a-b)=(c+d)\times(c-d)$, that is, $a^2-b^2=c^2-d^2$, then will b be greater than d; for a in the equation $a^2-b^2=c^2-d^2$ write $c+k$, and there reſults $2ck+k^2=b^2-d^2$, whence b^2-d^2 is an affirmative quantity, and conſequently b greater than d.

Ex. 4. Let, as in Ex. 1. the ratio $a+b:b$ be greater than $c+d:d$, then will $b:a-b$ be leſs than the ratio $d:c-d$. By the preceding method tranſlate theſe ratios into the two equations $\frac{b}{a+b}+k=\frac{d}{c+d}$ and $\frac{a-b}{b}=\frac{c-d}{d}+k'$, reduce theſe equations, ſo as to take away their denominators, and there reſult $bc+bd+\overline{a+b}\times\overline{c+d}k=ad+bd$ and $da-db=bc-bd+bdk'$, and conſequently $k=\frac{ad-bc}{(a+b)(c+d)}$ and $k'=\frac{ad-bc}{bd}$; but theſe two fractions which expreſs the values of k and k' have the ſame numerators, and their denominators both affirmative; therefore, if one k be affirmative, the other k' will alſo be affirmative.

Cor. From theſe principles can eaſily be deduced innumerable propoſitions of this ſort. Aſſume two or more ratios, of which let ſome be ſuppoſed greater than others; then, from the above-mentioned transformation, by addition, ſubtraction, multiplication, diviſion, &c. can be found ſuch functions of the

the above-mentioned quantities, that fome may become greater than others, and thence may be deduced the propofitions above-mentioned.

7. It may not be improper in this place to adjoin a few obfervations on finding the limits of fome quantities in which others contained in given equations become negative or affirmative.

1. Given an equation involving two unknown quantities x and y; the limits of the quantity y, between which the quantity x will become affirmative or negative, may be deduced from the following principles.

The quantity x paffes from affirmative to negative or from negative to affirmative, either through nothing or infinite; or from two impoffible roots it paffes to affirmative or negative through two or more equal roots; and, *vice verfâ*, from affirmative or negative to two or more impoffible roots through two or more equal roots.

Find therefore the values of y, when x becomes $=0$, or infinite; and alfo all the cafes in which two, &c. values of x become equal, that is, when its roots become impoffible; and from thence can be deduced the limits of the quantity y, between which (x) becomes affirmative or negative.

2. If $x=\frac{P}{Q}$ be an affirmative quantity, then P will be affirmative or negative, according as Q is an affirmative or negative quantity, &c. Affume therefore $P=0$ and $Q=0$, and from the roots of the refulting equation can be deduced the cafes, in which (x) becomes an affirmative quantity.

3. If more (n) unknown quantities (x, y, z, v, &c.) be contained in a given equation; then, by the preceding method, find the limits of (z, v, &c.), between which (x) becomes an affirmative or negative quantity, and let the quantities denoting the limits contain not more than ($n-1$) unknown quantities: from

from the above-mentioned quantities or equations expreffing the limits, find others denoting their limits, which do not contain more than $(n-2)$ above-mentioned quantities, and fo on.

4. Often from the fubftitution of the limits of given quantities can be acquired the limits of the remaining one (x). Find all the greateft values of the quantity (x) contained between the above-mentioned limits, and thence can be deduced the limits fought.

5. If there are given (m) equations involvin g$(m+1)$ or more unknown quantities; then fometimes with, and fometimes without, reducing them to others involving more few unknown quantities can be found by the preceding method limits; and from comparing the limits fo acquired can fometimes be deduced the limits fought.

6. If a given function of the unknown quantities $(x, y, z,$ &c.) is afferted to be contained between given limits, when other functions of the above-mentioned quantities are contained between given limits, and the demonftration of it is required; from the given equations and the given functions find limits of the unknown quantities refpectively, and if the latter limits are contained between the former, the propofition is generally true, otherwife not.

7. From the above-mentioned principles can be found the cafes in which an unknown quantity (x) admits of one or more affirmative values.

8. It appears from the principles before delivered, that the finding the number of affirmative and negative roots of a given equation neceffarily includes the finding the number of its impoffible roots; and therefore it may not be improper to fubjoin fomewhat on what has been done on this fubject.

1. DESCARTES gave a method of finding the number of affirmative and negative roots of a given equation, when all its roots are poſſible; but all the roots are very ſeldom in equations of ſuperior dimenſions poſſible, unleſs when the equation is purpoſedly made.

2. It has been demonſtrated by others and myſelf, that the equation will at leaſt have ſo many changes of ſigns from + to −, and − to +, as there are affirmative roots, and ſo many continued progreſſes from + to + and − to −, as there are negative roots.

3. A rule for finding in general the number of affirmative or negative roots in a biquadratic, and in the equation $x^n + Ax^m + B = 0$, was firſt publiſhed in the Medit. Algebr.

4. HARRIOT demonſtrated a method of finding the number of impoſſible roots contained in a cubic equation. In the year 1757 I ſent to the Royal Society a method of finding the number of impoſſible roots contained in a biquadratic and quadrato-cubic equations, and in the equation $x^n \pm Ax^m \pm B = 0$.

5. SCHOOTEN gave a method of finding the number of impoſſible roots which can be concluded from the deficient terms of an equation. NEWTON gave a rule which often diſcovers the number of impoſſible roots contained in a given equation. CAMPBELL diſcovered a new rule on the ſame ſubject. Mr. MACLAURIN has added ſomewhat more general on theſe ſubjects: theſe rules may be rendered more general by a principle firſt given in the Miſcell. Analyt. *viz.* multiplying the given equation into a quantity $x - a$ or $(x - a) \times (x - b)$, &c. and finding from the rule the number of impoſſible roots contained in the given equation. Similar and more general rules and principles have been added in the Medit. Algebr. Theſe rules, in equations of ſuperior dimenſions, ſeldom diſcover the true number of impoſſible

poſſible roots. I believe alſo, that I firſt gave a rule in the Miſcell. Analyt. for finding the number of impoſſible roots from finding an equation, whoſe roots are the ſquares &c. of the roots of a given equation, which rule in equations of ſuperior dimenſions ſometimes finds impoſſible roots, when NEWTON's, CAMPBELL's, &c. rules fail, and fails when they find them; and alſo a rule for finding impoſſible roots from an equation, whoſe roots are the ſquares of the differences of the roots of the given equation; this rule (as has been obſerved by me in the Miſcell. Analyt. and Philoſophical Tranſactions) always diſcovers whether all the roots of the given equation are poſſible or not; and the laſt term of the reſulting equation diſcovers alſo, whether 0, 4, 8, 12, &c. or 2, 6, 10, 14, &c. impoſſible roots, are contained in the given equation; to which may be ſubjoined, if the given equation has r poſſible and $n-r=2t$ impoſſible roots, that the number of changes of ſigns from + to − and − to + in the reſulting equation will not be leſs than $r \cdot \frac{r-1}{2}$, and the number of continued progreſſes from + to + and − to − will not be leſs than t: whence, if the number of continued progreſſes be t', the number of impoſſible roots will not be greater than $2t'$, and the number of poſſible roots not leſs than $n-2t'$. If the number of changes of ſigns be h', the number of poſſible roots will not be greater than r', where $r' \times \frac{r'-1}{2}$ is the greateſt poſſible number which does not exceed h', and the number of impoſſible roots not leſs than $n-r'$. Another rule was, I believe, firſt given by me in the Miſcell. Analyt. 1762, for finding impoſſible roots by finding an equation whoſe roots are z, where $x^n - px^{n-1} + qx^{n-2} - \&c. = z$, and $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-2} - \&c. = 0$.

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In the Medit. Algebr. ſomewhat has been added concerning impoſſible, affirmative, and negative values of the unknown quantities in an equation which involves two or more unknown quantities; and alſo was firſt delivered a rule from the number of affirmative, negative, and impoſſible roots of an equation being known to find the number of impoſſible, negative, and affirmative roots of an equation, whoſe roots have a given algebraical relation to the roots of a given equation; on which two laſt ſubjects little, I believe, had been before publiſhed.